James Ruse Agricultural High School
Year 12 2009 Term 1 Extension 1

Question 1.

(a) Find A if $\sum_{k=1}^{4} k\ln k = \ln A$

(b) (i) Express $1.\dot{7}\dot{2}$ as an infinite series.

(ii) Hence write $1.\dot{7}\dot{2}$ as an equivalent fraction in simplest terms.

(c) Find $\frac{d}{dx}\left(\operatorname{cosec}\left(x^2+1\right)\right)$.

(d) Find $\int \cos x\, e^{\sin x}\,dx$.

(e) (i) Write $3\sqrt{3}\sin x - 3\cos x$ in the form $R\cos(x-\alpha)$,where $R > 0$ and $0 < \alpha < 2\pi$.

(ii) Find the first value of $x > 0$ when $3\sqrt{3}\sin x - 3\cos x$ is a maximum.

Question 2.

(a) Differentiate with respect to x in simplest terms **:**

(i) $\frac{\tan 3x}{\sin 3x}$.

(ii) $e^{5x}\ln x^2$.

(b) Find $\int \frac{9+x}{4+x^2}\,dx$.

(c) (i) Simplify $\frac{9}{2x+1} + \frac{4}{3x+4}$.

(ii) Hence evaluate $\int_0^1 \frac{7x+8}{6x^2+11x+4}\,dx$.

Question 3.

(a) Find the area bounded by the curve $y = \tan x$, the x axis and the line $x = \frac{\pi}{3}$.

(b) If the sum S_n of n terms of a sequence is given by the formula $S_n = 8n^2 - 6n$, find the formula for the *nth* term T_n of the sequence.

(c) Find the formula S_n for the summation of n terms of the sequence **:**

$$2\frac{1}{2} + 4\frac{1}{4} + 6\frac{1}{8} + 8\frac{1}{16} + \ldots\ldots..$$

(d) (i) Sketch the graph of $y = \frac{3x+4}{2x-1}$.

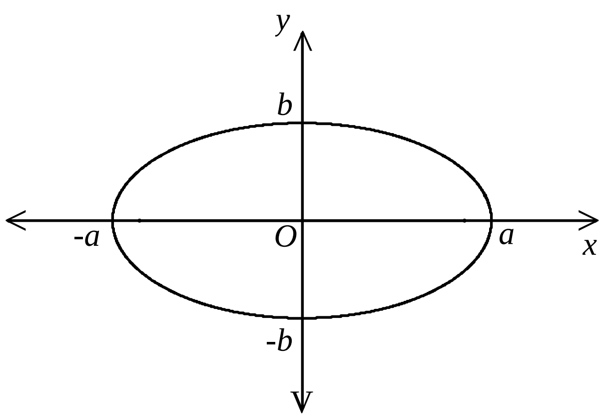
(ii) Find the values of x for a sum to infinity to exist **:**

$$1 + \left(\frac{3x+4}{2x-1}\right) + \left(\frac{3x+4}{2x-1}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots..$$

Question 4.

(a) (i) Use the substitution $x = r\sin\theta$ and evaluate $\int_0^r \sqrt{r^2 - x^2}\, dx$, where r is a constant.

(ii) An ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is shown below.



Using (i) find the area of the ellipse.

(b) A man invests \$180 000 at 4% p.a. at the beginning of the month.
Interest is credited at the end of each month.
If the man withdraws \$ 2000 at the end of each month, find the number of \$ 2000 withdrawals that can be made.

Question 5.

(a) Given $y = \cos\left(x^2\right)$ in the domain $0 \le x \le \pi$ then **:**

(i) Show $\dfrac{d^2y}{dx^2} = -2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right)$

(ii) Find the location of the turning points, and determine their nature.

(iii) Graph $y = \cos\left(x^2\right)$ in the domain $0 \le x \le \pi$.

(b) A helicopter which is initially on the ground, rises vertically at a constant speed of 5*m/s.*
If a man is initially 50 metres horizontally from the helicopter, find the rate of change of the angle of elevation θ of the helicopter when the helicopter is 20 metres above the ground.

Question 6.

(a) Without using Calculus and using a scale of 1cm = 1 unit on each axis, graph the following functions on the same axes in the domain $0 < x < 2\pi$ **:**

(i) $y = \cos x$

(ii) $y = \cos 2x$

(iii) $y = \cos x + \cos 2x$

(b) Find the volume of revolution when the region bounded by the curve $y = \ln\left(x+1\right)$,
The y axis and the line $y = \ln 2$ is rotated around the y axis.

(c) Prove by Mathematical Induction **:**

$$1\times 2^2 + 2\times 3^2 + 3\times 4^2 + \dots\dots\dots\dots\dots\dots n\left(n+1\right)^2 = \frac{n}{12}\left(n+1\right)\left(n+2\right)\left(3n+5\right)$$

End of Exam

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Q1

(a) $\sum_{k=1}^{4} k\ln k = 1\ln 1 + 2\ln 2 + 3\ln 3 + 4\ln 4$

$= \ln 2^2 . 3^3 . 4^4$

$= \ln A$

$A = 2^2 . 3^3 . 4^4$

$= 27648$

(b) (i) $1\dot{7}\dot{2} = 1 + \frac{72}{100} + \frac{72}{10000} + \ ---$

(ii) $= 1 + \dfrac{\frac{72}{100}}{1 - \frac{1}{100}}$

$= 1 + \frac{72}{99}$

$= 1\frac{8}{11}$

(c) $\frac{d}{dx} \operatorname{cosec}(x^2+1) = -2x \operatorname{cosec}(x^2+1)\cot(x^2+1)$

(d) $\int \cos x \, e^{\sin x} dx = e^{\sin x} + C$

(e) (i) $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\sin\alpha = 3\sqrt{3}$ $\quad R > 0 \quad \frac{\pi}{2} < \alpha < \pi$.

$R\cos\alpha = -3$

$R = \sqrt{(3\sqrt{3})^2 + 3^2}$

$= \sqrt{36}$

$= 6$

$\tan\alpha = -\sqrt{3}$

$\alpha = \frac{2\pi}{3}$

$\therefore \ 3\sqrt{3}\sin x - 3\cos x = 6\left(\cos\left(x - \frac{2\pi}{3}\right)\right)$

(ii) $x - \frac{2\pi}{3} = 0$

$x = \frac{2\pi}{3}$

Q2 (a) (i) $\frac{d}{dx}\left(\frac{\tan 3x}{\sin 3x}\right) = \frac{d}{dx}(\sec 3x)$

$= 3\sec 3x\tan 3x$

(ii) $\frac{d}{dx} e^{5x}\ln x^2 = \frac{d}{dx}\left(2e^{5x}\ln x\right)$

$= 2\frac{e^{5x}}{x} + 10e^{5x}\ln x$

$= 2e^{5x}\left[\frac{1}{x} + 5\ln x\right]$

(b) $\int \frac{9+x}{4+x^2} dx = \int \left(\frac{9}{4+x^2} + \frac{x}{4+x^2}\right) dx$

$= \frac{9}{2}\tan^{-1}\frac{x}{2} + \frac{1}{2}\ln(x^2+4) + C$

(c) (i) $\frac{9}{2x+1} + \frac{4}{3x+4} = \frac{9(3x+4) + 4(2x+1)}{(2x+1)(3x+4)}$

$= \frac{35x + 40}{6x^2 + 11x + 4}$

(ii) $\int_0^1 \frac{7x+8}{6x^2+11x+4} dx = \frac{1}{5}\int_0^1 \left(\frac{9}{2x+1} + \frac{4}{3x+4}\right) dx$

$= \frac{1}{5}\left[\frac{9}{2}\ln(2x+1) + \frac{4}{3}\ln(3x+4)\right]_0^1$ as for $x > 0$

$= \frac{1}{5}\left[\frac{9}{2}\ln\left(\frac{3}{1}\right) + \frac{4}{3}\ln\left(\frac{7}{4}\right)\right]$

Q3 (c) Area $= \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$

$= \left[-\ln\cos x\right]_0^{\frac{\pi}{3}}$ $\quad \cos x > 0$ for $0 < x < \frac{\pi}{3}$

$= -\left[\ln\frac{1}{2} - 0\right] = \ln 2$ square units

b) $S_n = 8n^2 - 6n$

$S_{n-1} = 8(n-1)^2 - 6(n-1)$

$T_n = S_n - S_{n-1}$

$= 16n - 14$

c) $S_n = 2\frac{1}{2} + 4\frac{1}{4} + 6\frac{1}{8} + 8\frac{1}{16}$

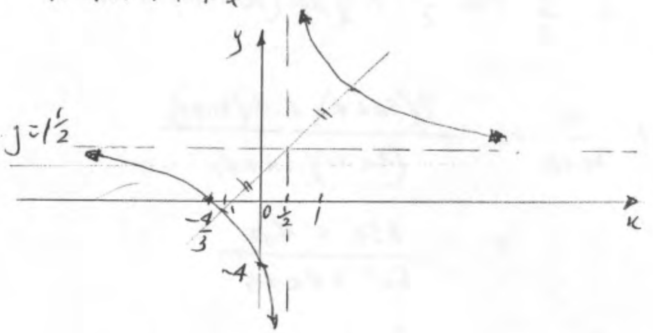
$= (2 + 4 + \dots 2n) + \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \frac{1}{2^n}\right)$

$= \frac{n}{2}\left[4 + (n-1).2\right] + \frac{1}{2}\left(\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right)$

$= n(n+1) + 1 - \left(\frac{1}{2}\right)^n$

$S_n = n^2 + n + 1 + 2^{-n}$

d(i)



(ii) For limiting sum $-1 < r < 1$ $r \neq 0$.

$\therefore \frac{3x+4}{2x-1} = 1$ $\qquad$ $\frac{3x+4}{2x-1} = -1$

$x = -5$ $\qquad$ $x = -\frac{3}{5}$

$\therefore$ Soln $\left\{-5 < x < -\frac{4}{3}\right\}$ OR $\left\{-\frac{4}{3} < x < -\frac{3}{5}\right\}$

Q4 (i) $\int_0^r \sqrt{r^2 - x^2}\, dx$ $\qquad$ $x = r\sin\theta$, $dx = r\cos\theta\, d\theta$, $x=0$ $\theta = 0$, $x = r$ $\theta = \frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\, .\, r\cos\theta\, d\theta$ (1)

$= \int_0^{\frac{\pi}{2}} \sqrt{r^2(1 - \sin^2\theta)}\, .\, r\cos\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{r^2\cos^2\theta}\, .\, r\cos\theta\, d\theta$

$= r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$

$= r^2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$ (1)

$= \frac{r^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{r^2}{2}\left[\frac{\pi}{2} + 0 - 0\right]$ (1)

$= \frac{\pi r^2}{4}$

(ii) Area ellipse $\qquad$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y = b\sqrt{1 - \frac{x^2}{a^2}}$ $= \frac{b}{a}\sqrt{a^2 - x^2}$

$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$

$= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$ (1)

$= 4\frac{b}{a} \cdot \frac{\pi . a^2}{4} = \pi ab$ (1)

Q5 (5)

(a)(i) $y = \cos x^2$

$\frac{dy}{dx} = -2x \sin x^2$

$\frac{d^2y}{dx^2} = -2x \cdot \cos x^2 \cdot 2x - 2\sin x^2$

$= -2\sin(x^2) - 4x^2\cos x^2$ (2)

(ii) For turning points

$\frac{dy}{dx} = 0$

$-2x\sin x^2 = 0$

$x = 0 \quad x^2 = 0, \pi, 2\pi, 3\pi \ldots$

$x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \pm\sqrt{3\pi}, \ldots$

But $x = \sqrt{\pi}$, $\sqrt{2\pi}$ or $\sqrt{3\pi}$ as $0 < x < \pi$

$y = -1 \quad y = 1 \quad y = -1$

Turning points $(\sqrt{\pi}, -1)$, $(\sqrt{2\pi}, 1)$ & $(\sqrt{3\pi}, -1)$ (1)

For nature of turning points test $\frac{d^2y}{dx^2}$ for sign concavity as $0 < x < \pi$ is continuous.

At $x = \sqrt{\pi}$ $\quad \frac{d^2y}{dx^2} = -2 \cdot 0 - 4\pi \times -1$

$= 4\pi$

> 0

$\therefore$ Relative minimum at $(\sqrt{\pi}, -1)$ (3)

At $x = \sqrt{2\pi}$ $\quad \frac{d^2y}{dx^2} = -2 \cdot 0 - 4 \cdot 2\pi \cdot 1$

$= -8\pi$

< 0

$\therefore$ Relative maximum at $(\sqrt{2\pi}, 1)$

At $x = \sqrt{3\pi}$ $\quad \frac{d^2y}{dx^2} = -2 \cdot 0 - 4 \cdot 3\pi \cdot -1$

$= 12\pi$

> 0

$\therefore$ Relative minimum at $(\sqrt{3\pi}, -1)$

monthly interest$_r$ = $\frac{1}{3}\%$ (6)

$r = \frac{1}{300}$

(b) Amount left end 1st month $= (180000)(1+r) - 2000$ (1)

Amount left end 2nd Month $= [180000(1+r) - 2000](1+r) - 2000$

$= 180000(1+r)^2 - 2000[1 + (1+r)]$

Amount left end 3rd Month $= [180000(1+r)^2 - 2000(1+(1+r))](1+r) - 2000$

$= (180000)(1+r)^3 - 2000[1 + (1+r) + (1+r)^3]$

Amount left end n months $= 180000 \cdot (1+r)^n - 2000[1 + (1+r) + \ldots (1+r)^n]$ (1)

$= 180000(1+r)^n - 2000\left[1 \cdot \frac{(1+r)^n - 1}{1+r-1}\right]$

$= 180000(1+r)^n - 2000\left[\frac{(1+r)^n - 1}{r}\right]$ (1)

$0 = 180000(1+r)^n - 2000\left[\frac{(1+r)^n - 1}{r}\right]$

$\therefore 180000\left(1+\frac{1}{300}\right)^n = 2000(300)\left[\left(1+\frac{1}{300}\right)^n - 1\right]$

$420000\left(\frac{301}{300}\right)^n = 600000$

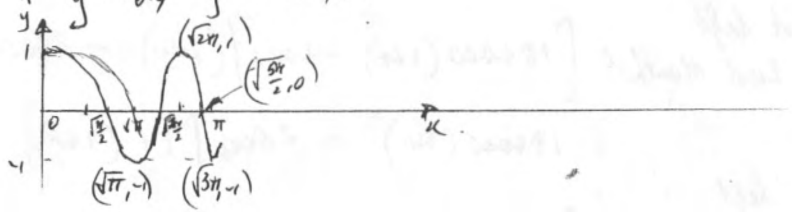
$\left(\frac{301}{300}\right)^n = \frac{10}{7}$

$n = 107.18$ (1)

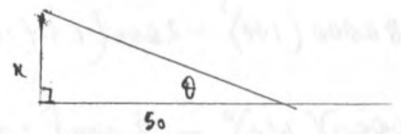
$\therefore$ Number of \$2000 payments is 107

(iii) $x=0 \quad y=1 \quad y'=0$

$x=\pi \quad y=-0.9 \quad y'=-2.70.$



(b)



$\tan\theta = \frac{x}{50}$

$\theta = \tan^{-1}\frac{x}{50}$

$\frac{d\theta}{dx} = \frac{50}{2500+x^2}$

$\frac{dx}{dt} = 5$

$\therefore$ Rate of change angle elevation $= \frac{d\theta}{dt}$

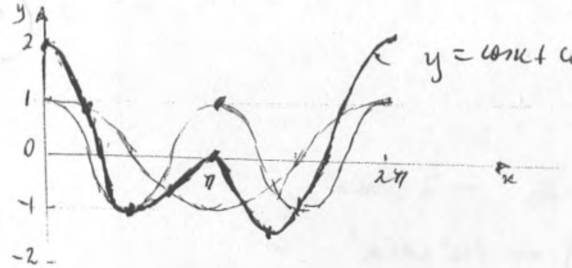
$= \frac{d\theta}{dx} \cdot \frac{dx}{dt}$ (1)

$= \frac{50}{2500+20^2} \cdot 5$ when $x=20$ (1)

$= \frac{250}{2900}$

$= \frac{5}{58}$ rad/s. (1)

Q6 (a)



(1) each graph

(b) $y = \ln(x+1)$

$x = e^y - 1$

Volume $V = \pi \int_a^b x^2\, dy$

$= \pi \int_0^{\ln 2} (e^y - 1)^2\, dy$ (1)

$= \pi \int_0^{\ln 2} (e^{2y} - 2e^y + 1)\, dy$

$= \pi \left[\frac{e^{2y}}{2} - 2e^y + y \right]_0^{\ln 2}$ (1)

$= \pi \left[2 - 4 + \ln 2 - \left(\frac{1}{2} - 2 + 0 \right) \right]$

$= \pi \left[\ln 2 - \frac{1}{2} \right]$ unit^3. (1)

6(c)

Step 1 $n=1$

$T_1 = 1.(1+1)^2$ $S_1 = \frac{1}{12}(1+1)(1+2)(3+5)$

$= 4$ $= \frac{2.3.8}{12}$

$= \frac{48}{12}$

$= 4$

$\therefore T_1 = S_1$

$\therefore$ statement true $n=1$

Step 2 Assume statement is true $n=k$

ie $1\times 2^2 + 2\times 3^2 + \ldots \quad k(k+1)^2 = \frac{k}{12}(k+1)(k+2)(3k+5)$

To prove statement is true $n=k+1$

ie $1\times 2^2 + 2\times 3^2 + \ldots \quad k(k+1)^2 + (k+1)(k+2)^2 = \frac{(k+1)}{12}(k+2)(k+3)(3k+8)$

Now $1\times 2^2 + 2\times 3^2 + \ldots \quad k(k+1)^2 + (k+1)(k+2)^2$

$= \frac{k}{12}(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$ By assumption

$= \frac{(k+1)}{12}(k+2)\left[3k^2+5k + 12(k+2)\right]$

$= \frac{(k+1)}{12}(k+2)\left[3k^2+17k+24\right]$

$= \frac{(k+1)}{12}(k+2)\left[3k^2+8k+9k+24\right]$

$= \frac{(k+1)}{12}(k+2)\left[k\left[3k+8\right]+3\left[3k+8\right]\right]$

$= \frac{(k+1)(k+2)(k+3)(3k+8)}{12}$

$\therefore$ If statement true $n=k$ it is also true $n=k+1$

Since statement is true $n=1$ it is also true $n=1+1=2$, $n=2+1=3$